

Global solutions to random 3D vorticity equations for small initial data

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Abstract

One proves the existence and uniqueness in $(L^p(\mathbb{R}^3))^3$, $\frac{3}{2} < p < 2$, of a global mild solution to random vorticity equations associated to stochastic 3D Navier-Stokes equations with linear multiplicative Gaussian noise of convolution type, for sufficiently small initial vorticity. This resembles some earlier deterministic results of T. Kato [15] and are obtained by treating the equation in vorticity form and reducing the latter to a random nonlinear parabolic equation. The solution has maximal regularity in the spatial variables and is weakly continuous in $(L^3 \cap L^{\frac{3p}{4p-6}})^3$ with respect to the time variable. Furthermore, we obtain the pathwise continuous dependence of solutions with respect to the initial data.

Keywords: stochastic Navier-Stokes equation, vorticity, Biot-Savart operator.

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1 Introduction

Consider the stochastic 3D Navier-Stokes equation

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$$\begin{aligned}
dX - \Delta X \, dt + (X \cdot \nabla)X \, dt &= \sum_{i=1}^N (B_i(X) + \lambda_i X) d\beta_i(t) + \nabla \pi \, dt \\
\nabla \cdot X &= 0 \\
X(0) &= x
\end{aligned}
\begin{aligned}
&\text{on } (0, \infty) \times \mathbb{R}^3, \\
&\text{on } (0, \infty) \times \mathbb{R}^3, \\
&\text{in } (L^p(\mathbb{R}^3))^3,
\end{aligned} \tag{1.1}$$

where $\lambda_i \in \mathbb{R}$, $x : \Omega \rightarrow \mathbb{R}^3$ is a random variable. Here π denotes the pressure and $\{\beta_i\}_{i=1}^N$ is a system of independent Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $(\mathcal{F}_t)_{t \geq 0}$, x is \mathcal{F}_0 -measurable, and B_i are the convolution operators

$$B_i(X)(\xi) = \int_{\mathbb{R}^3} h_i(\xi - \bar{\xi}) X(\bar{\xi}) d\bar{\xi} = (h_i * X)(\xi), \quad \xi \in \mathbb{R}^3, \tag{1.2}$$

where $h_i \in L^1(\mathbb{R}^3)$, $i = 1, 2, \dots, N$, and Δ is the Laplacian on $(L^2(\mathbb{R}^3))^3$.

It is not known whether (1.1) has a probabilistically strong solution in the mild sense for all time. Therefore, we shall rewrite (1.1) in vorticity form and then transform it into a random PDE, which we shall prove, has a global in time solution for \mathbb{P} -a.e. fixed $\omega \in \Omega$, provided the initial condition is small enough.

Consider the vorticity field

$$U = \nabla \times X = \text{curl } X \tag{1.3}$$

and apply the curl operator to equation (1.1). We obtain (see e.g. [4], [8]) the transport vorticity equation

$$\begin{aligned}
dU - \Delta U \, dt + ((X \cdot \nabla)U - (U \cdot \nabla)X) \, dt &= \sum_{i=1}^N (h_i * U + \lambda_i U) d\beta_i \\
&\text{in } (0, \infty) \times \mathbb{R}^3,
\end{aligned} \tag{1.4}$$

$$U(0, \xi) = U_0(\xi) = (\text{curl } x)(\xi), \quad \xi \in \mathbb{R}^3.$$

The vorticity U is related to the velocity X by the equation

$$X(t, \xi) = K(U(t))(\xi), \quad t \in (0, \infty), \quad \xi \in \mathbb{R}^3, \tag{1.5}$$

where K is the Biot–Savart integral operator

$$K(u)(\xi) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\xi - \bar{\xi}}{|\xi - \bar{\xi}|^3} \times u(\bar{\xi}) d\bar{\xi}, \quad \xi \in \mathbb{R}^3. \tag{1.6}$$

Then one can rewrite the vorticity equation (1.4) as

$$\begin{aligned} dU - \Delta U dt + ((K(U) \cdot \nabla)U - (U \cdot \nabla)K(U))dt \\ = \sum_{i=1}^N (h_i * U + \lambda_i U) d\beta_i \quad \text{in } (0, \infty) \times \mathbb{R}^3, \\ U(0, \xi) = U_0(\xi), \quad \xi \in \mathbb{R}^3. \end{aligned} \quad (1.7)$$

Equivalently,

$$\begin{aligned} U(t) = e^{t\Delta} U_0 - \int_0^t e^{(t-s)\Delta} ((K(U(s)) \cdot \nabla)U(s) - (U(s) \cdot \nabla)K(U(s))) ds \\ + \int_0^t \sum_{i=1}^N e^{-(t-s)\Delta} (h_i * U(s) + \lambda_i U(s)) d\beta_i(s), \quad t \geq 0. \end{aligned} \quad (1.8)$$

Now, we consider the transformation

$$U(t) = \Gamma(t)y(t), \quad t \in [0, \infty), \quad (1.9)$$

where $\Gamma(t) : (L^2(\mathbb{R}^3))^3 \rightarrow (L^2(\mathbb{R}^3))^3$ is the linear continuous operator defined by the equations

$$d\Gamma(t) = \sum_{i=1}^N (B_i + \lambda_i I) \Gamma(t) d\beta_i(t), \quad t \geq 0, \quad \Gamma(0) = I, \quad (1.10)$$

where (see (1.2))

$$B_i u = h_i * u, \quad \forall u \in (L^p(\mathbb{R}^3))^3, \quad i = 1, \dots, N, \quad p \in (1, \infty). \quad (1.11)$$

We also set

$$\tilde{B}_i = B_i + \lambda_i I, \quad i = 1, \dots, N, \quad (1.12)$$

where I is the identity operator.

Since $B_i B_j = B_j B_i$, equation (1.10) has a solution Γ and can be equivalently expressed as (see [9], Section 7.4)

$$\Gamma(t) = \prod_{i=1}^N \exp \left(\beta_i(t) \tilde{B}_i - \frac{t}{2} \tilde{B}_i^2 \right), \quad t \geq 0. \quad (1.13)$$

Here (1.10) is meant in the sense that, for every $z_0 \in (L^2(\mathbb{R}^3))^3$, the continuous (\mathcal{F}_t) -adapted $(L^2(\mathbb{R}^3))^3$ -valued process $z(t) := \Gamma(t)z_0$, $t \geq 0$, solves the following SDE on $H := (L^2(\mathbb{R}^3))^3$,

$$dz(t) = \sum_{i=1}^N \tilde{B}_i z(t) d\beta_i(t), \quad z(0) = z_0,$$

where H is equipped with the usual scalar product $\langle \cdot, \cdot \rangle$.

Applying the Itô formula in (1.7) (the justification for this is as in [2]), we obtain for y the random differential equation

$$\begin{aligned} \frac{dy}{dt}(t) - \Gamma^{-1}(t)\Delta(\Gamma(t)y(t)) + \Gamma^{-1}(t)(K(\Gamma(t)y(t)) \cdot \nabla)(\Gamma(t)y(t)) \\ - (\Gamma(t)y(t) \cdot \nabla)(K(\Gamma(t)y(t))) = 0, \quad t \in [0, \infty), \\ y(0) = U_0. \end{aligned} \quad (1.14)$$

Taking into account that, for all i , $B_i \Delta = \Delta B_i$ on $H^2(\mathbb{R}^3)$, it follows by (1.10), (1.13) that $\Delta \Gamma(t) = \Gamma(t) \Delta$ on $H^2(\mathbb{R}^3)$, $\forall t \geq 0$.

In what follows, equation (1.14) will be taken in the following mild sense

$$y(t) = e^{t\Delta}U_0 + \int_0^t e^{(t-s)\Delta}\Gamma^{-1}(s)M(\Gamma(s)y(s))ds, \quad t \in [0, \infty), \quad (1.15)$$

where

$$(e^{t\Delta}u)(\xi) = \frac{1}{(4\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \exp\left(-\frac{|\xi - \bar{\xi}|^2}{4t}\right) u(\bar{\xi}) d\bar{\xi}, \quad t \in [0, \infty), \quad \xi \in \mathbb{R}^3, \quad (1.16)$$

and M is defined by

$$M(u) = -[(K(u) \cdot \nabla)(u) - (u \cdot \nabla)(K(u))], \quad t \in [0, \infty). \quad (1.17)$$

We note that $U(t) = \Gamma(t)y(t)$ is the solution to the equation

$$U(t) = e^{t\Delta}\Gamma(t)U_0 + \int_0^t e^{(t-s)\Delta}\Gamma(t)\Gamma^{-1}(s)M(U(s))ds, \quad (1.18)$$

which may be viewed as the random version of the stochastic vorticity equation (1.8).

Our aim here and the principal contribution of this work is to show that, for every $\varepsilon \in (0, 1)$, there exists $\Omega_\varepsilon \in \mathcal{F}$ such that $\mathbb{P}(\Omega_\varepsilon) \geq 1 - \varepsilon$ and, for all $\omega \in \Omega_\varepsilon$, we have the existence and uniqueness of a solution (in the mild sense) for (1.15) if the vorticity of x , i.e., $U_0 = \text{curl } x$, is \mathbb{P} -a.s. sufficiently small in a sense to be made precise in Theorem 1.1 below. We recall that, for a deterministic Navier–Stokes equation, such a result was first established by T. Kato [15] (see also T. Kato and H. Fujita [16]) and extended later to more general initial data by Y. Giga and T. Miyakawa [14], M. Taylor [21], H. Koch and D. Tataru [17]. However, the standard approach [15], [16] cannot be applied in the present case for one reason: the nonlinear inertial term $(X \cdot \nabla)X$ cannot be conveniently estimated in the space $C_b([0, \infty); L^p(\Omega \times \mathbb{R}^d))$ and similarly for the nonlinearity arising in (1.7). As regards the stochastic 3D Navier-Stokes equations, to best of our knowledge all global existence results were limited to martingale solutions. Since the fundamental work [11], the literature on (global) martingale solutions for stochastic 3D-Navier-Stokes equations has grown enormously. We refer, e.g., to [6], [10], [12], [13], [18], and the references therein.

In the following, we denote by L^p , $1 \leq p \leq \infty$, the space $(L^p(\mathbb{R}^3))^3$ with the norm $|\cdot|_p$, by $W^{1,p}$ the corresponding Sobolev space and by $C_b([0, \infty); L^p)$ the space of all bounded and continuous functions $u : [0, \infty) \rightarrow L^p$ with the sup norm. We also set $D_i = \frac{\partial}{\partial \xi_i}$, $i = 1, 2, 3$, and denote by $\nabla \cdot u$ the divergence of u , while

$$((u \cdot \nabla)v)_j = u_i D_i v_j, \quad j = 1, 2, 3, \quad u = \{u_i\}_{i=1}^3, \quad v = \{v_j\}_{j=1}^3.$$

As usual

$$q' = \frac{q}{q-1} \quad \text{for } q \in (1, \infty).$$

We set for $p \in (\frac{3}{2}, 3)$

$$\eta(t) = \|\Gamma(t)\|_{L(L^p, L^p)} \|\Gamma(t)\|_{L(L^{\frac{3p}{3-p}}, L^{\frac{3p}{3-p}})} \|\Gamma^{-1}(t)\|_{L^q, L^q}, \quad t \geq 0, \quad (1.19)$$

where for $q \in (1, \infty)$, $\|\cdot\|_{L(L^q, L^q)}$ is the norm of the space $L(L^q, L^q)$ of linear continuous operators on L^q .

For $p \in [1, \infty)$, we denote by \mathcal{Z}_p the space of all functions $y : (0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that

$$\begin{aligned} t^{1-\frac{3}{2p}} y &\in C_b([0, \infty); L^p), \\ t^{\frac{3}{2}(1-\frac{1}{p})} D_i y &\in C_b([0, \infty); L^p), \quad i = 1, 2, 3. \end{aligned} \quad (1.20)$$

The space \mathcal{Z}_p is endowed with the norm

$$\|y\|_{p,\infty} = \sup \left\{ t^{1-\frac{3}{2p}} |y(t)|_p + t^{\frac{3}{2}(1-\frac{1}{p})} |D_i y(t)|_p; t \in (0, \infty), i = 1, 2, 3 \right\}. \quad (1.21)$$

In the following, we take $\lambda_i \in \mathbb{R}$ such that

$$|\lambda_i| > (\sqrt{12} + 3)|h_i|_1, \quad \forall i = 1, 2, \dots, N. \quad (1.22)$$

We note that

$$\|B_i\|_{L(L^q, L^q)} \leq |h_i|_{L^1}, \quad \forall i = 1, \dots, N.$$

Theorem 1.1 is the main result.

Theorem 1.1. *Let $p, q \in (1, \infty)$ such that*

$$\frac{3}{2} < p < 2, \quad \frac{1}{q} = \frac{2}{p} - \frac{1}{3}. \quad (1.23)$$

Let $\Omega_0 = \{\sup_{t \geq 0} \eta(t) < \infty\}$ and consider (1.15) for fixed $\omega \in \Omega_0$. Set $\Gamma(t) := \Gamma(t)(\omega)$, $\eta(t) := \eta(t, \omega)$. Then $\mathbb{P}(\Omega_0) = 1$ and there is a positive constant C^ independent of $\omega \in \Omega_0$ such that, if $U_0 \in L^{\frac{3}{2}}$ is such that*

$$\sup_{t \geq 0} \eta(t) |U_0|_{\frac{3}{2}} \leq C^*, \quad (1.24)$$

then the random equation (1.15) has a unique solution $y \in \mathcal{Z}_p$ which satisfies

$$M(\Gamma(t)y) \in L^1(0, T; L^q). \quad (1.25)$$

Moreover, for each $\varphi \in L^3 \cap L^{q'}$, the function $t \rightarrow \int_{\mathbb{R}^3} y(t, \xi) \varphi(\xi) d\xi$ is continuous on $[0, \infty)$. The map $U_0 \rightarrow y$ is Lipschitz from $L^{\frac{3}{2}}$ to \mathcal{Z}_p .

In particular, the random vorticity equation (1.18) has a unique solution U such that $\Gamma^{-1}U \in \mathcal{Z}_p$.

Remark 1.2. Concerning condition (1.24), we note that an elementary calculation shows that

$$\eta(t) \leq \prod_{i=1}^N \exp(3|\beta_i(t)|(|h_i|_1 + |\lambda_i|) - t\alpha_i), \quad t \in [0, \infty),$$

where $\alpha_i := \frac{1}{2} \lambda_i^2 - \frac{3}{2} (|h_i|_1^2 + 2|\lambda_i| |h_i|_1)$, which is strictly positive by (1.22).

By the law of the iterated logarithm, it follows that

$$\sup_{t \geq 0} \eta(t) < \infty, \quad \mathbb{P}\text{-a.e.},$$

hence for $\Omega_r := \{\sup_{t \geq 0} \eta(t) \leq r\}$ we have $\mathbb{P}(\Omega_r^c) \rightarrow 0$ as $r \rightarrow \infty$.

But, taking into account that, for each $r > 0$ and all $\nu > 0$, $i = 1, \dots, N$, we have (see Lemma 3.4 in [1])

$$\mathbb{P} \left[\sup_{t \geq 0} \{\exp(\beta_i(t) - \nu t)\} \geq r \right] = r^{-2\nu},$$

and, more explicitly, we get that

$$\mathbb{P}(\Omega_r^c) \leq 2Nr^{-\frac{N\alpha}{\gamma^2}}, \quad \forall r > 0,$$

where $\alpha = \min_{1 \leq i \leq N} \alpha_i$, $\gamma = 3 \max\{(|h_i|_1 + |\lambda_i|); i \leq N\}$. Therefore, if $\omega \in \Omega_r$ and $U_0 = U_0(\omega)$ satisfies

$$|U_0|_{\frac{3}{2}} \leq \frac{C^*}{r}, \tag{1.26}$$

then condition (1.24) holds. It is trivial to define such an \mathcal{F} -measurable function $U_0 : \Omega \rightarrow L^{\frac{3}{2}}$, for instance,

$$U_0 := \sum_{n=1}^{\infty} \frac{C^*}{n} \mathbf{1}_{\{n-1 \leq \sup_{t \geq 0} \eta(t) < n\}} u_0,$$

for some $u_0 \in L^{\frac{3}{2}}$. But, of course, U_0 is not \mathcal{F}_0 -measurable and so the process $U(t)$, $t \geq 0$, given by Theorem 1.1, is not $(\mathcal{F}_t)_{t \geq 0}$ -adapted. Therefore, $U = \Gamma(t)y$ is not a solution to the stochastic vorticity equation (1.8). However, it can be viewed as a generalized solution to (1.8).

It should also be mentioned that assumption (1.22) is not necessary for existence of a solution to equation (1.15), but only to make sure that condition (1.24) is not void.

2 Proof of Theorem 1.1

To begin with, we note below in Lemma 2.1 a few immediate properties of the operator Γ defined in (1.10)–(1.13).

Lemma 2.1. *We have*

$$|\Gamma(t)z|_q + |\Gamma^{-1}(t)z|_q \leq C_t |z|_q, \quad t \in [0, \infty), \quad \forall z \in L^q, \quad \forall q \in [1, \infty), \quad (2.1)$$

and

$$|\nabla(\Gamma(t)z)|_q \leq \|\Gamma(t)\|_{L(L^q, L^q)} |\nabla z|_q, \quad \text{for all } z \in W^{1,q}(\mathbb{R}^3). \quad (2.2)$$

Proof. By (1.2), (1.11) and by the Young inequality, we see that

$$|B_i(u)|_q \leq (|h_i|_1 + |\lambda_i|) |u|_q, \quad \forall u \in L^q, \quad i = 1, \dots, N. \quad (2.3)$$

Recalling (1.13), we see by (2.3) that (2.1), (2.2) hold, as claimed. \blacksquare

Lemma 2.2. *Let $\frac{1}{q} = \frac{1}{r_1} + \frac{1}{r_2}$, $\frac{3}{2} < r_1 < \infty$, $r_1^* = \frac{3r_1}{3+r_1}$, $1 < q < \infty$. Then, for some $C > 0$ independent of ω ,*

$$|M(\Gamma(t)z)|_q \leq C \|\Gamma(t)\|_{L(L^{r_1}, L^{r_1})} \|\Gamma(t)\|_{L(L^{r_2}, L^{r_2})} (|z|_{r_1} |z|_{r_2} + |z|_{r_1^*} |\nabla z|_{r_2}), \quad (2.4)$$

$$t \in [0, \infty),$$

for all $z \in L^{r_1} \cap L^{r_2} \cap L^{r_1^*}$ with $\nabla z \in L^{r_2}$.

Proof. We have by (1.17) and (2.1)

$$|M(\Gamma(t)z)|_q \leq |(K(\Gamma(t)z) \cdot \nabla)(\Gamma(t)z)|_q + |(\Gamma(t)z \cdot \nabla)K(\Gamma(t)z)|_q. \quad (2.5)$$

On the other hand, by (2.2) and the Hölder inequality we have

$$\begin{aligned} |(K(\Gamma(t)z) \cdot \nabla)(\Gamma(t)z)|_q &\leq |K(\Gamma(t)z)|_{r_1} |\nabla(\Gamma(t)z)|_{r_2} \\ &\leq \|\Gamma(t)\|_{L(L^{r_1}, L^{r_1})} \|\Gamma(t)\|_{L(L^{r_2}, L^{r_2})} |K(z)|_{r_1} |\nabla z|_{r_2}. \end{aligned} \quad (2.6)$$

Now, we recall the classical estimate for Riesz potentials (see [20], p. 119)

$$\left| \int_{\mathbb{R}^3} \frac{f(\bar{\xi})}{|\xi - \bar{\xi}|^2} d\xi \right|_{\beta} \leq C |f|_{\alpha}, \quad \forall f \in L^{\alpha},$$

where $\frac{1}{\beta} = \frac{1}{\alpha} - \frac{1}{3}$, $\alpha \in (1, 3)$. By virtue of (1.6), this yields

$$|K(u)|_{\beta} \leq C |u|_{\alpha}; \quad \forall u \in L^{\alpha}, \quad \frac{1}{\beta} = \frac{1}{\alpha} - \frac{1}{3}, \quad (2.7)$$

and so, for $\beta = r_1$, $\alpha = \frac{3r_1}{3+r_1} = r_1^*$, we get by (2.2) and (2.6) the estimate

$$|(K(\Gamma(t)z) \cdot \nabla)(\Gamma(t)z)|_q \leq C \|\Gamma(t)\|_{L(L^{r_1}, L^{r_1})} \|\Gamma(t)\|_{L(L^{r_2}, L^{r_2})} |z|_{r_1^*} |\nabla z|_{r_2}. \quad (2.8)$$

(Here and everywhere in the following, $|\nabla z|_p$ means $\sup\{|D_i z|_p; i = 1, 2, 3\}$.) Taking into account that, by the Calderon–Zygmund inequality (see [7], Theorem 1),

$$|\nabla K(z)|_{\tilde{p}} \leq C|z|_{\tilde{p}}, \quad \forall z \in L^{\tilde{p}}, \quad 1 \leq \tilde{p} < \infty, \quad (2.9)$$

one obtains that

$$|(\Gamma(t)z \cdot \nabla)(K(t)\Gamma(t)z)|_q \leq C\|\Gamma(t)\|_{L(L^{r_1}, L^{r_1})}\|\Gamma(t)\|_{L(L^{r_2}, L^{r_2})}|z|_{r_1}|z|_{r_2}. \quad (2.10)$$

Substituting (2.8), (2.10) in (2.5), one obtains estimate (2.4), as claimed. ■

Lemma 2.3. *Let $r_1 = \frac{3r_2}{3-r_2}$, $\frac{3}{2} < r_2 < 3$, $q = \frac{3r_1}{r_1+6}$. Then, we have, for some $C > 0$ independent of ω ,*

$$|M(\Gamma(t)z)|_q \leq C\|\Gamma(t)\|_{L(L^{r_1}, L^{r_1})}\|\Gamma(t)\|_{L(L^{r_2}, L^{r_2})}|z|_{r_2}|\nabla z|_{r_2}, \quad \forall z \in W^{1, r_2}. \quad (2.11)$$

Proof. We have by the Sobolev–Gagliardo–Nirenberg inequality (see, e.g., [5], p. 278)

$$|z|_{r_1} \leq C|\nabla z|_{r_2}, \quad \forall z \in W^{1, r_2}(\mathbb{R}^3).$$

Substituting in (2.4) and taking into account that $r_1^* = r_2$, we obtain (2.11), as claimed. ■

In the following, we fix $p = r_2$, r_1 and q as in Lemma 2.3, (2.11), that is,

$$\frac{3}{2} < p < 2, \quad r_1 = \frac{3p}{3-p}, \quad \frac{1}{q} = \frac{2}{p} - \frac{1}{3}. \quad (2.12)$$

We write equation (1.15) as

$$y(t) = G(y)(t) = e^{t\Delta}U_0 + F(y)(t), \quad t \in [0, \infty), \quad (2.13)$$

where

$$F(z)(t) = \int_0^t e^{(t-s)\Delta}\Gamma^{-1}(s)M(\Gamma(s)z(s))ds, \quad t \in [0, \infty). \quad (2.14)$$

By (1.16), we have for $1 < \tilde{q} \leq \tilde{p} < \infty$ the estimates

$$|e^{t\Delta}u|_{\tilde{p}} \leq Ct^{-\frac{3}{2}(\frac{1}{\tilde{q}} - \frac{1}{\tilde{p}})}|u|_{\tilde{q}}, \quad u \in L^{\tilde{q}}, \quad (2.15)$$

$$|D_j e^{t\Delta}u|_{\tilde{p}} \leq Ct^{-\frac{3}{2}(\frac{1}{\tilde{q}} - \frac{1}{\tilde{p}}) - \frac{1}{2}}|u|_{\tilde{q}}, \quad j = 1, 2, 3. \quad (2.16)$$

(Everywhere in the following, we shall denote by C several positive constants independent of ω and $t \geq 0$.)

We apply (2.15), with $\tilde{q} = q$, $\tilde{p} = p$. By (2.11)–(2.14), we obtain that

$$\begin{aligned} |F(z(t))|_p &\leq C \int_0^t (t-s)^{-\frac{1}{2}(\frac{3}{p}-1)} |\Gamma^{-1}(s)M(\Gamma(s)z(s))|_q ds \\ &\leq C \int_0^t (t-s)^{-\frac{1}{2}(\frac{3}{p}-1)} \|\Gamma(s)\|_{L(L^{\frac{3p}{3-p}}, L^{\frac{3p}{3-p}})} \\ &\quad \|\Gamma(s)\|_{L(L^p, L^p)} \|\Gamma^{-1}(s)\|_{L(L^q, L^q)} |z(s)|_p |\nabla z(s)|_p ds. \end{aligned} \quad (2.17)$$

Similarly, we obtain by (2.16) that

$$\begin{aligned} |D_j F(z(t))|_p &\leq C \int_0^t (t-s)^{-\frac{3}{2p}} \|\Gamma(s)\|_{L(L^{\frac{3p}{3-p}}, L^{\frac{3p}{3-p}})} \\ &\quad \|\Gamma(s)\|_{L(L^p, L^p)} \|\Gamma^{-1}(s)\|_{L(L^q, L^q)} |z(s)|_p |\nabla z(s)|_p ds, \quad j = 1, 2, 3. \end{aligned} \quad (2.18)$$

We consider the Banach space \mathcal{Z}_p defined by (1.20), that is,

$$\mathcal{Z}_p = \left\{ y; \ t^{1-\frac{3}{2p}} y \in C_b([0, \infty); L^p), \ t^{\frac{3}{2}(1-\frac{1}{p})} D_j y \in C_b([0, \infty); L^p), \right. \\ \left. j = 1, 2, 3 \right\}, \quad (2.19)$$

with the norm

$$\|z\|_{p, \infty} = \|z\| = \sup_{t>0} \left\{ \left(t^{1-\frac{3}{2p}} |z(t)|_p + t^{\frac{3}{2}(1-\frac{1}{p})} |D_i z(t)|_p \right), \ i = 1, 2, 3 \right\}. \quad (2.20)$$

We note that

$$|z(t)|_p |\nabla z(t)|_p \leq C t^{-\frac{5}{2}+\frac{3}{p}} \|z\|^2, \quad \forall z \in \mathcal{Z}_p, \ t \in (0, \infty). \quad (2.21)$$

By (2.17) and (2.20) we see that, for $z \in \mathcal{Z}_p$, we have

$$\begin{aligned} |F(z(t))|_p &\leq \int_0^t (t-s)^{-\frac{1}{2}(\frac{3}{p}-1)} \|\Gamma(s)\|_{L(L^{\frac{3p}{3-p}}, L^{\frac{3p}{3-p}})} \|\Gamma(s)\|_{L(L^p, L^p)} \\ &\quad \|\Gamma^{-1}(s)\|_{L(L^q, L^q)} |z(s)|_p |\nabla z(s)|_p ds \\ &\leq C \int_0^t (t-s)^{-\frac{1}{2}(\frac{3}{p}-1)} |s|^{-\frac{5}{2}+\frac{3}{p}} \|\Gamma(s)\|_{L(L^{\frac{3p}{3-p}}, L^{\frac{3p}{3-p}})} \\ &\quad \|\Gamma(s)\|_{L(L^p, L^p)} \|\Gamma^{-1}(s)\|_{L(L^q, L^q)} ds \|z\|^2 \\ &\leq C t^{\frac{3}{2p}-1} \sup_{0 \leq s \leq t} \eta(s) \int_0^1 (1-s)^{-\frac{1}{2}(\frac{3}{p}-1)} s^{-\frac{5}{2}+\frac{3}{p}} ds \|z\|^2, \quad \forall t > 0, \end{aligned} \quad (2.22)$$

where η is given by (1.19). This yields

$$t^{1-\frac{3}{2p}}|F(z(t))|_p \leq C \sup_{0 \leq s \leq t} \{\eta(s)\} B\left(\frac{3}{2}\left(\frac{2}{p}-1\right), \frac{3}{2}\left(1-\frac{1}{p}\right)\right) \|z\|^2, \quad \forall t > 0, \quad (2.23)$$

where B is the classical beta function (which is finite by virtue of (1.23)).

Similarly, by (2.16) and (2.21), we have, for $j = 1, 2, 3$,

$$\begin{aligned} |D_j F(z)(t)|_p &\leq C \int_0^t (t-s)^{-\frac{3}{2p}} s^{-\frac{5}{2}+\frac{3}{p}} \|\Gamma(s)\|_{L(L^{\frac{3p}{3-p}}, L^{\frac{3p}{3-p}})} \\ &\quad \|\Gamma(s)\|_{L(L^p, L^p)} \|\Gamma^{-1}(s)\|_{L(L^q, L^q)} ds \|z\|^2 \\ &\leq C \sup_{0 \leq s \leq t} \{\eta(s)\} t^{-\frac{3}{2}(1-\frac{1}{p})} B\left(3\left(\frac{1}{p}-\frac{1}{2}\right), 1-\frac{3}{2p}\right) \|z\|^2, \quad t > 0. \end{aligned} \quad (2.24)$$

Hence,

$$t^{\frac{3}{2}(1-\frac{1}{p})} |D_j F(z(t))|_p \leq C \sup_{0 \leq s \leq t} \eta(s) \|z\|^2, \quad \forall z \in \mathcal{Z}_p, \quad t > 0, \quad j = 1, 2, 3. \quad (2.25)$$

By (2.15)–(2.16), we have

$$\begin{aligned} |e^{t\Delta} U_0|_p &\leq C t^{\frac{3}{2p}-1} |U_0|_{\frac{3}{2}}, \quad t > 0, \\ |D_j e^{t\Delta} U_0|_p &\leq C t^{\frac{3}{2p}-\frac{3}{2}} |U_0|_{\frac{3}{2}}, \quad t > 0, \quad j = 1, 2, 3. \end{aligned}$$

Therefore, by (2.20) we get

$$\|e^{t\Delta} U_0\| \leq C |U_0|_{\frac{3}{2}}. \quad (2.26)$$

By (2.20), (2.23), (2.25), (2.26), we get,

$$\|G(z)\| \leq C_1 \left(|U_0|_{\frac{3}{2}} + \sup_{t \geq 0} \eta(t) \|z\|^2 \right), \quad \forall z \in \mathcal{Z}_p, \quad (2.27)$$

where $C_1 > 0$ is independent of ω .

We set

$$\eta_\infty = \sup_{t \geq 0} \eta(t), \quad (2.28)$$

and so (2.27) yields

$$\|G(z)\| \leq C_1 (|U_0|_{\frac{3}{2}} + \eta_\infty \|z\|^2), \quad \forall z \in \mathcal{Z}_p. \quad (2.29)$$

We set

$$\Sigma = \{z \in \mathcal{Z}_p; \|z\| \leq R^*\}$$

and note that, by (2.29), it follows that $G(\Sigma) \subset \Sigma$ if

$$|U_0|_{\frac{3}{2}} \eta_\infty \leq (4C_1^2)^{-1}, \quad (2.30)$$

(so U_0 must depend on ω) and $R^* = R^*(\omega)$ is given by

$$R^* = 2C_1 |U_0|_{\frac{3}{2}}. \quad (2.31)$$

(We recall that C_1 is independent of ω and U_0 .) Moreover, by (1.17) and (2.14), we have, for all $z, \bar{z} \in \mathcal{Z}_p$,

$$\begin{aligned} G(z)(t) - G(\bar{z})(t) = & - \int_0^t e^{(t-s)\Delta} \Gamma^{-1}(s) [(K\Gamma(s)(z(s) - \bar{z}(s)) \cdot \nabla) \Gamma(s)z(s) \\ & + (K(\Gamma(s)\bar{z}(s)) \cdot \nabla) \Gamma(s)(z(s) - \bar{z}(s)) - \Gamma(s)(z(s) - \bar{z}(s)) \cdot \nabla K(\Gamma(s)z(s)) \\ & - (\Gamma(s)\bar{z}(s) \cdot \nabla) K(\Gamma(s)(z(s) - \bar{z}(s)))] ds. \end{aligned}$$

Proceeding as above, we get, as in (2.17), (2.22), (2.23), that

$$\begin{aligned} & |G(z)(t) - G(\bar{z})(t)|_p \\ & \leq C \int_0^t (t-s)^{-\frac{1}{2}(\frac{3}{p}-1)} \|\Gamma(s)\|_{L(L^{\frac{3p}{p-3}}, L^{\frac{3p}{p-3}})} \|\Gamma(s)\|_{L(L^p, L^p)} \\ & \quad \|\Gamma^{-1}(s)\|_{L(L^q, L^q)} (|z(s) - \bar{z}(s)|_p (|\nabla z(s)|_p + |\nabla \bar{z}(s)|_p) \\ & \quad + |\nabla z(s) - \nabla \bar{z}(s)|_p (|z(s)|_p + |\bar{z}(s)|_p)) ds \\ & \leq Ct^{-(1-\frac{3}{2p})} \sup_{0 \leq s \leq t} \eta(s) \|z - \bar{z}\| (\|z\| + \|\bar{z}\|), \quad \forall t > 0, \end{aligned} \quad (2.32)$$

and also (see (2.18), (2.24), (2.25))

$$|D_j G(z)(t) - D_j G(\bar{z}(t))|_p \leq Ct^{-\frac{3}{2}(1-\frac{1}{p})} \sup_{0 \leq s \leq t} \eta(s) \|z - \bar{z}\| (\|z\| + \|\bar{z}\|), \quad \forall t > 0,$$

for $j = 1, 2, 3$. Hence, by (2.20) and (2.28), we obtain that

$$\|G(z) - G(\bar{z})\| \leq C_2 \eta_\infty R^* \|z - \bar{z}\|, \quad \forall z, \bar{z} \in \Sigma, \quad (2.33)$$

where C_2 is independent of ω .

Then, by (2.31), (2.33), it follows that, if (2.30) and

$$2C_1C_2\eta_\infty|U_0|_{\frac{3}{2}} < 1, \quad (2.34)$$

hold, then the operator G is a contraction on Σ and so there is a unique solution $U \in \Sigma$ to (1.15) provided (1.24) holds with $C^* < (2C_1C_2)^{-1}$.

Now, as seen earlier, by (2.11), (1.15) and (2.21) we have

$$\begin{aligned} |M(\Gamma(t)y(t))|_q &\leq C\|\Gamma(t)\|_{L(L^{\frac{3p}{3-p}}, L^{\frac{3p}{3-p}})}\|\Gamma(t)\|_{L(L^p, L^p)}|y(t)|_p|\nabla y(t)|_p \\ &\leq C\|\Gamma(t)\|_{L(L^{\frac{3p}{3-p}}, L^{\frac{3p}{3-p}})}\|\Gamma(t)\|_{L(L^p, L^p)}t^{-\frac{5}{2}+\frac{3}{p}}\|y\|^2, \quad \forall t > 0. \end{aligned} \quad (2.35)$$

On the other hand, we have for all $\varphi \in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned} \int_{\mathbb{R}^3} y(t, \xi) \cdot \varphi(\xi) d\xi &= \int_{\mathbb{R}^3} (e^{t\Delta})U_0(\xi) \cdot \varphi(\xi) d\xi \\ &\quad + \int_0^t \int_{\mathbb{R}^3} \Gamma^{-1}(s)M(\Gamma(s)y(s)) \cdot e^{(t-s)\Delta}\varphi(\xi) d\xi ds. \end{aligned} \quad (2.36)$$

Recalling that, for all $1 \leq \tilde{p} < \infty$, $|e^{t\Delta}\varphi|_{\tilde{p}} \leq |\varphi|_{\tilde{p}}$, it follows by (2.35) that

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}^3} \Gamma^{-1}(s)M(\Gamma(s)y(s)) \cdot e^{(t-s)\Delta}\varphi(\xi) d\xi ds \right| \\ \leq C \sup_{0 \leq s \leq t} \eta(s) \int_0^t s^{-\frac{5}{2}+\frac{3}{p}} ds \|y\|^2 |\varphi|_{q'} \\ \leq C \sup_{0 \leq s \leq t} \eta(s) t^{\frac{3}{p}-\frac{3}{2}} \|y\|^2 |\varphi|_{q'}, \quad \forall t \in (0, \infty). \end{aligned} \quad (2.37)$$

We also have by (2.26)

$$\left| \int_{\mathbb{R}^3} e^{t\Delta}U_0(\xi)\varphi(\xi) d\xi \right| \leq C|U_0|_{\frac{3}{2}}|\varphi|_3, \quad \forall t \in [0, \infty).$$

Combining the latter with (2.36), (2.37), we obtain that, for $T > 0$,

$$\left| \int_{\mathbb{R}^3} y(t, \xi) \cdot \varphi(\xi) d\xi \right| \leq CT^{\frac{3}{p}-\frac{3}{2}}(|\varphi|_{q'} + |\varphi|_3), \quad \forall \varphi \in L^{q'} \cap L^3, \quad t \in [0, T].$$

Hence, by (2.36) and since $t \rightarrow e^{t\Delta}U_0$ is continuous on $L^{\frac{3}{2}}$, the function $t \rightarrow y(t)$ is $L^3 \cap L^{q'}$ weakly continuous on $[0, \infty)$, where $q' = \frac{3p}{4p-6}$.

If $U = y(t, U_0) \in \mathcal{Z}_p$ is the solution to (1.15), equivalently (2.13), we have for all U_0, \bar{U}_0 satisfying (1.24) (see (2.26) and (2.33))

$$\begin{aligned} \|y(\cdot, U_0) - y(\cdot, \bar{U}_0)\| &\leq \|e^{t\Delta}(U_0 - \bar{U}_0)\| + \|F(y(\cdot, U_0)) - F(y(\cdot, \bar{U}_0))\| \\ &\leq C|U_0 - \bar{U}_0|_{\frac{3}{2}} + \eta_\infty R^* C_2 \|y(\cdot, U_0) - y(\cdot, \bar{U}_0)\|. \end{aligned}$$

Recalling that by (2.31) and (2.34) we have $R^* C_2 \eta_\infty < 1$, this yields

$$\|y(\cdot, U_0) - y(\cdot, \bar{U}_0)\| \leq \frac{C}{1 - R^* C_2 \eta_\infty} |U_0 - \bar{U}_0|_{\frac{3}{2}} \leq C(\omega) |U_0 - \bar{U}_0|_{\frac{3}{2}},$$

and so, the map $y \rightarrow U(\cdot, U_0)$ is Lipschitz from $L^{\frac{3}{2}}$ to \mathcal{Z}_p . This completes the proof of Theorem 1.1. \blacksquare

It should be noted that, by (2.30) and (2.31), we have by the Fernique theorem

$$|U_0|_{\frac{3}{2}}, R^* \in \bigcap_{r \geq 1} L^r(\Omega),$$

and so, taking into account that $y \in \Sigma$, we see by (2.19), (2.20) that

$$\sup_{t \geq 0} \left\{ t^{1 - \frac{3}{2p}} |y(t)|_p + t^{\frac{3}{2}(1 - \frac{1}{p})} |D_i y(t)|_p \right\} \in \bigcap_{r \geq 1} L^r(\Omega), \quad i = 1, 2, 3. \quad (2.38)$$

We have, therefore, the following completion of Theorem 1.1.

Corollary 2.4. *Under the assumptions of Theorem 1.1, the solution $y = y(t, \omega)$ to the equation (1.15) satisfies (2.38). The same result holds for the solution $U(t) = \Gamma(t)y(t)$ of the random vorticity equation (1.18).*

3 The random version of the 3D Navier-Stokes equation

We fix in (1.1) the initial random variable x by the formula

$$x = K(U_0), \quad (3.1)$$

where $U_0 = \text{curl } x$, $U_0 = U_0(\omega)$ satisfies condition (1.24) for all $\omega \in \Omega_0$ (see Remark 1.2). If y is the corresponding solution to equation (1.15) given by Theorem 1.1, we define the process X by formula (1.5), that is,

$$X(t) = K(U(t)) = K(\Gamma(t)y(t)), \quad \forall t \in [0, \infty). \quad (3.2)$$

By (2.7), where U is the solution to the vorticity equation (1.1), we have

$$|X(t)|_{\frac{3p}{3-p}} \leq C|U(t)|_p, \quad \forall t \in [0, \infty). \quad (3.3)$$

(Everywhere in the following, C are positive constants independent of $\omega \in \Omega$.)

On the other hand, by the Carlderone–Zygmund inequality (2.9), we have

$$|D_i X(t)|_p \leq C|U(t)|_p, \quad i = 1, 2, 3. \quad (3.4)$$

By (3.3) and by Theorem 1.1, it follows that

$$t^{1-\frac{3}{2p}} X \in C_b([0, \infty); L^{\frac{3p}{3-p}}), \quad (3.5)$$

while, by (3.4), we have for $i = 1, 2, 3$

$$t^{\frac{3}{2}(1-\frac{1}{p})} D_i X \in C_b([0, \infty); L^p). \quad (3.6)$$

Now, if in (2.9) we take $z = D_j X$, we get that, for all $i, j = 1, 2, 3$,

$$t^{\frac{3}{2}(1-\frac{1}{p})} |D_i D_j X|_p \leq C t^{\frac{3}{2}(1-\frac{1}{p})} |D_i U|_p \leq C, \quad \forall t \in [0, \infty).$$

This yields

$$t^{\frac{3}{2}(1-\frac{1}{p})} D_{ij}^2 X \in C_b([0, \infty); L^p), \quad i, j = 1, 2, 3. \quad (3.7)$$

Moreover, by Corollary 2.4, we also have

$$t^{1-\frac{3}{2p}} X \in C_b([0, \infty); L^r(\Omega; L^{\frac{3p}{3-p}})), \quad \forall r \geq 1, \quad (3.8)$$

$$t^{\frac{3}{2}(1-\frac{1}{p})} D_i X \in C_b([0, \infty); L^r(\Omega; L^p)), \quad i = 1, 2, 3, \quad (3.9)$$

$$t^{\frac{3}{2}(1-\frac{1}{p})} D_{ij} X \in C_b([0, \infty); L^r(\Omega; L^p)), \quad i, j = 1, 2, 3. \quad (3.10)$$

Now, if in equation (1.18) one applies the Biot–Savart operator K , we obtain for X the equation

$$X(t) = K(e^{t\Delta} \Gamma(t) \operatorname{curl} x) + \int_0^t K(e^{(t-s)\Delta} \Gamma(t) \Gamma^{-1}(s) M(\operatorname{curl} X(s))) ds, \quad (3.11)$$

$t \geq 0,$

where M is given by (1.17). It should be noted that, by virtue of (3.7)–(3.10), the right hand side of (3.11) is well defined.

Equation (3.11) can be viewed as the random version of the Navier-Stokes equation (1.1). However, since, as seen earlier, U_0 is not \mathcal{F}_0 -measurable, the processes $t \rightarrow y(t)$, $t \rightarrow U(t)$ are not $(\mathcal{F}_t)_{t \geq 0}$ -adapted, and so X is not $(\mathcal{F}_t)_{t \geq 0}$ -adapted, too. Therefore, (3.11) cannot be transformed back into (1.1). By Theorem 1.1, it follows that

Theorem 3.1. *Under assumptions (1.24), the random Navier-Stokes equation (3.11) has a unique solution X satisfying (3.7)-(3.10).*

Remark 3.2. As easily seen from the proofs, Theorem 1.1 extends mutatis mutandis to the noises $\sum_{i=1}^N \sigma_i(t, X) \dot{\beta}_i(t)$, where

$$\sigma_i(t, x)(\xi) = \int_{\mathbb{R}^3} h_i(t, \xi - \bar{\xi}) x(\bar{\xi}) d\bar{\xi}, \quad \xi \in \mathbb{R}^3, \quad i = 1, \dots, N,$$

where $t \rightarrow h_i(t, \xi)$ is continuous and

$$|h_i(t)|_1 \leq C, \quad \forall t \geq 0, \quad i = 1, \dots, N.$$

Remark 3.3. The linear multiplicative case $B_i(X) := \alpha_i X$, $i = 1, \dots, N$, that is $h_i := \delta$, where δ is the Dirac measure, can be approximated by taking $h_i(\xi) = \frac{1}{\varepsilon^d} \rho\left(\frac{\xi}{\varepsilon}\right)$, where $\rho \in C_0^\infty(\mathbb{R}^d)$, support $\rho \subset \{\xi; |\xi|_d \leq 1\}$, $\int \rho(\xi) d\xi = 1$.

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